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STATISTICAL CHARACTERISTICS OF THE DIFFUSION
OF A CHEMICALLY ACTIVE ADDITIVE
IN A TURBULENT MIXING ZONE

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In the article a numerical solution of the connected system of the equations of turbulent transfer for the fields of the velocity and concentration of a chemically active additive is used to calculate a number of the second moments of the concentration field in a flat mixing zone. The system of transfer equations is derived from the equations for a common function of the distribution of the fields of the pulsations of the velocity and the concentration [1] and is simplified in the approximation of the boundary layer. A closed form of the transfer equations is obtained on the level of three moments, using the hypothesis of four moments [2] and its generalized form for mixed moments of the field of the velocity and the field of a passive scalar. The differential operator of the closed system of the equations of turbulent transfer for the fields of the velocity and the concentration is found by a method of closure not of the parabolic type but of a weakly hyperbolic type [3]. An implicit difference scheme proposed in [4] is used for the numerical solution. The results of the numerical solution are compared with the experimental data of [5].

1. System of Equations for the Moments of the Field
of the Concentration

The turbulent diffusion of a dynamically passive additive in a free inhomogeneous turbulent flow of an incompressible liquid is considered in an Euler description. The dynamic passivity of the additive postulates that the field of the velocity $u(x)$ does not undergo any appreciable effect from the side of the process of turbulent diffusion of the additive. The additive can react chemically with the medium of the flow. The chemical reaction under consideration can be assumed to be passive, which can be regarded as justified for "weak" chemical reactions in the flow (taking place relatively slowly and quietly) and small concentrations of the impurity. It is assumed that the fields of the pulsations of the velocity and concentration of the additive can be described by a common distribution function, satisfying some "kinetic" equation [1]. The equations for the moments of the field of the concentration in a free inhomogeneous turbulent flow are de-

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rived in a manner analogous to that used in [1, 4] to obtain equations for the moments of the field of the velocity. It is necessary only to adopt the relationship for the value of $d\theta/dt$ entering into the equation for the common distribution function. This quantity consists of two terms, the first of which models the rate of change of the concentration of the additive in an element of the liquid as a result of molecular diffusion, and the second, the rate of the chemical reaction. The first term, whose form is determined from semiempirical considerations, can also be interpreted as the scalar dissipation or as the rate of mixing of the substance of the additive up to molecular scales. The expression for $d\theta/dt$ has the form

$$\frac{d\theta}{dt} = -3 \frac{b_0}{L} E^{1/2} \theta' + \mu \theta^n, \quad (1.1)$$

where θ' is the fluctuation of the concentration in an Euler system of coordinates; b_0 is an empirical constant; μ is the "rate constant of the chemical reaction" ($\mu = \text{const}$, $\mu \geq 0$); n is a whole positive number ($n \geq 1$), and is the "order" of the chemical reaction. (Here and in what follows, notation not specially stipulated corresponds to the notation of [4].) An expression for the scalar dissipation [the first term in (1.1)], similar to that adopted here, was also used, for example, in [6]. Approximation of the rate of the chemical reaction [the second term in (1.1)] in the form of a power function only of the concentration $\theta(\mathbf{x})$ (but not of \mathbf{x}) relates to the case of small concentrations of the additive in comparison with the concentrations of the surrounding medium of the flow. The chemical reaction with $n=1$ takes place in accordance with a first order. If $n > 1$ (for example, for $n=2$, i.e., "a second-order reaction"), then in the equation for the variance of the fluctuations of the field of the concentrations the term describing the rate of the chemical transformation and, in the case of a reaction of arbitrary order, having the form $2\mu \langle \theta^{n+1} \rangle$, contains with $n=2$, in addition to the second moment, also the third moment $\langle \theta^3 \rangle$. For this term a determining equation must be written in which fourth moments appear. They can be (approximately) expressed in terms of second moments, using the generalized hypothesis of four moments. In a first consideration, it is expedient to limit ourselves to a chemical reaction of the "first order," assuming $n=1$ in (1.1).

Below, equations are written for moments of the first three orders of the field of the concentration, relating to the fully developed free turbulent flow of a completely turbulent incompressible liquid, under steady-state external conditions without a pressure gradient, where the additive reacts chemically with the medium of the flow in accordance with a "first order."

The equation of conservation of the mean concentration of the additive

$$\frac{\partial}{\partial x_k} [\langle u_k \rangle \langle \theta \rangle + \langle u_k' \theta' \rangle] = \mu \langle \theta \rangle; \quad (1.2)$$

the equation for the components of the vector of the turbulent flow of the additive

$$\langle u_k \rangle \frac{\partial \langle u_\alpha' \theta' \rangle}{\partial x_k} = - \langle u_k' u_\alpha' \rangle \frac{\partial \langle \theta \rangle}{\partial x_k} - \langle u_k' \theta' \rangle \frac{\partial \langle u_\alpha \rangle}{\partial x_k} - \frac{\partial \langle u_k' u_\alpha' \theta' \rangle}{\partial x_k} - \langle u_\alpha' \theta' \rangle \frac{E^{1/2}}{L} \left[\frac{3}{8} (a_0 + 8b_0) + c_0 \right] + \mu \langle u_\alpha' \theta' \rangle; \quad (1.3)$$

the equation for the variance of the fluctuations of the field of the concentration

$$\langle u_k \rangle \frac{\partial \langle \theta'^2 \rangle}{\partial x_k} = - 2 \langle u_k' \theta' \rangle \frac{\partial \langle \theta \rangle}{\partial x_k} - \frac{\partial \langle u_k' \theta'^2 \rangle}{\partial x_k} - 6b_0 \frac{E^{1/2}}{L} \langle \theta'^2 \rangle + 2\mu \langle \theta'^2 \rangle; \quad (1.4)$$

the equations for the third mixed moments of the fields of the pulsations of the velocity and the concentration

$$\begin{aligned} \langle u_k \rangle \frac{\partial \langle u_\alpha' u_\beta' \theta' \rangle}{\partial x_k} &= - \langle u_k' u_\alpha' u_\beta' \rangle \frac{\partial \langle \theta \rangle}{\partial x_k} - \langle u_k' u_\alpha' \theta' \rangle \frac{\partial \langle u_\beta \rangle}{\partial x_k} \\ &- \langle u_k' u_\beta' \theta' \rangle \frac{\partial \langle u_\alpha \rangle}{\partial x_k} - \langle u_k' u_\alpha' \rangle \frac{\partial \langle u_\beta' \theta' \rangle}{\partial x_k} - \langle u_k' u_\beta' \rangle \frac{\partial \langle u_\alpha' \theta' \rangle}{\partial x_k} - \underbrace{\langle u_k' \theta' \rangle \frac{\partial \langle u_\alpha' u_\beta' \rangle}{\partial x_k}}_{(*)} \\ &- \frac{3}{4} \left[\frac{a_0 + 4b_0}{L} \right] E^{1/2} \langle u_\alpha' u_\beta' \theta' \rangle - \frac{c_0}{L} E^{1/2} \langle u_\alpha' u_\beta' \theta' \rangle + \mu \langle u_\alpha' u_\beta' \theta' \rangle; \end{aligned} \quad (1.5)$$

$$\begin{aligned} \langle u_k \rangle \frac{\partial \langle u_\alpha' \theta'^2 \rangle}{\partial x_k} &= - \langle u_k' u_\alpha' \rangle \frac{\partial \langle \theta'^2 \rangle}{\partial x_k} - 2 \langle u_k' \theta' \rangle \frac{\partial \langle u_\alpha' \theta' \rangle}{\partial x_k} \\ &- 2 \langle u_k' u_\alpha' \theta' \rangle \frac{\partial \langle \theta \rangle}{\partial x_k} - \frac{3}{8} \left[\frac{a_0 + 16b_0}{L} \right] E^{1/2} \langle u_\alpha' \theta'^2 \rangle - c_0 \frac{E}{L} \langle u_\alpha' \theta'^2 \rangle + \\ &+ 2\mu \langle u_\alpha' \theta'^2 \rangle; \end{aligned} \quad (1.6)$$

where a_0, b_0, c_0 are empirical constants; the angular brackets denote averaging with respect to the ensemble of the realizations. The fourth mixed moments, appearing in the equations of the third mixed moments (1.5), (1.6), are expressed in terms of second moments, using the hypothesis of fourth moments*

$$\begin{cases} \langle u'_k u'_\alpha u'_\beta \theta' \rangle = \langle u'_k u'_\alpha \rangle \langle u'_\beta \theta' \rangle + \langle u'_k u'_\beta \rangle \langle u'_\alpha \theta' \rangle + \langle u'_k \theta' \rangle \langle u'_\alpha u'_\beta \rangle, \\ \langle u'_k u'_\alpha \theta' \theta' \rangle = \langle u'_k u'_\alpha \rangle \langle \theta'^2 \rangle + 2 \langle u'_k \theta' \rangle \langle u'_\alpha \theta' \rangle. \end{cases} \quad (1.7)$$

Relationships (1.7), being exact for Gaussian distributions, in the general case can be regarded as some approximation. The use of relationships (1.7) to close the equations for the moments in free turbulent flows with a gradient of the mean velocity, although indirect, obviously makes it possible to form a judgment with respect to the noncontradictory nature of the hypothesis of fourth moments. It is shown in [8] that closing the equations for the moments of the field of a passive additive in a field of isotropic turbulence using noncumulative approximations (the infinitesimal level of which is the hypothesis of fourth moments with respect to the equality to zero of the fourth cumulants of the fields of the velocity and the concentration) leads to nonphysical results. In a model diffusion problem of the variance of the fluctuations of the concentration of an additive in a field of steady-state isotropic turbulence, the hypothesis of the zero of the fourth cumulants of the fields of the velocity and the concentration was used [8] to close the equations for the moments. It was found that, with large diffusion times, a positive quantity, i.e., the variance of the fluctuations of a scalar field, takes on negative values, contradicting its physical meaning.

In view of this it is considered that the value of the variance of the fluctuations of the concentration of an additive $\langle \theta'^2 \rangle$, positive everywhere in the region of integration, obtained as the result of a numerical solution of the closed system of equations of turbulent transfer for the fields of the velocity and concentration in a free inhomogeneous turbulent flow, can therefore argue in support of the proposition that the nonlinearity of the equations of the field of the velocity does not lead to the above-mentioned nonphysical results (numerical results relating to the function $\langle \theta'^2 \rangle$ are given below). It is possible that the postulated absence of nonphysical results for the moments of the field of the concentration in the mixing zone can be connected not only with the nonlinearity of the dynamically inhomogeneous turbulent flow, but also with the fact that the principal dynamic mechanism in inhomogeneous turbulence is the interaction between the mean flow and the turbulence, and not interaction between the fluctuational components of the turbulent flow, which, in isotropic turbulence, determines the whole dynamics [8].

An immediate result of the use of relationships (1.7) for closing the equations for the moments (1.2)-(1.6) is the following: the system (1.2)-(1.6) is found to be connected with the closed system of moment equations of the field of the velocity (system (1.1)-(1.6) from [4]) through the coefficients with the derivatives and source terms and through the derivatives of the second moments of the field of the velocity, due to the appearance, in the left-hand side of (1.5), of the term marked with an asterisk. It can be noted that this term is not small in comparison with the other terms, even in the approximation of the boundary layer, although a reliable evaluation of it is difficult due to the lack of experimental data.

2. Turbulent Diffusion of a Chemically Inert Additive in a Flat Mixing Zone

We consider the problem of the turbulent diffusion of a chemically inert ($\mu=0$) additive in a two-dimensional mixing zone, for which the system of equations (1.2)-(1.6) is simplified, in the approximation of the boundary layer, bringing in the experimental data of [5]. A schematic picture of the flow in the mixing zone and its geometric determination are given in [4]. The result of the simplification is the following system of five quasilinear equations for five sought functions:

$$\left. \begin{aligned} \langle u \rangle \frac{\partial \langle \theta \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle \theta \rangle}{\partial y} + \frac{\partial S}{\partial y} &= 0, \\ \langle u \rangle \frac{\partial S}{\partial x} + \langle v \rangle \frac{\partial S}{\partial y} + \langle v'^2 \rangle \frac{\partial \langle \theta \rangle}{\partial y} + \frac{\partial \langle S'v' \rangle}{\partial y} &= -\frac{3}{8} \left[\frac{a_0 + 8b_0}{L} \right] E^{1/2} S - (c_0/L) E^{1/2} S, \\ \langle u \rangle \frac{\partial \langle \theta'^2 \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle \theta'^2 \rangle}{\partial y} + 2S \frac{\partial \langle \theta \rangle}{\partial y} + \frac{\partial \langle \theta'^2 v' \rangle}{\partial y} &= -6b_0 (E^{1/2}/L) \langle \theta'^2 \rangle, \\ \langle u \rangle \frac{\partial \langle S'v' \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle S'v' \rangle}{\partial y} + 2 \langle v'^2 \rangle \frac{\partial S}{\partial y} + S \frac{\partial \langle v'^2 \rangle}{\partial y} &= -\frac{3}{4} \left[\frac{a_0 + 4b_0}{L} \right] E^{1/2} \langle S'v' \rangle - \frac{c_0}{L} E^{1/2} \langle S'v' \rangle, \end{aligned} \right\} \quad (2.1)$$

*The hypothesis of four moments in generalized form (1.7) was first used in the problem of the turbulent diffusion of an additive in a field of isotropic turbulence.

$$\langle u \rangle \frac{\partial \langle \theta'^2 v' \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle \theta'^2 v' \rangle}{\partial y} + 2S \frac{\partial S}{\partial y} + \langle v'^2 \rangle \frac{\partial \langle \theta'^2 \rangle}{\partial y} = -\frac{3}{8} \left[\frac{a_0 + 16b_0}{L} \right] E^{1/2} \langle \theta'^2 v' \rangle - \frac{c_0}{L} E^{1/2} \langle \theta'^2 v' \rangle, \quad \left| \right.$$

where

$$S \equiv \langle \theta' v' \rangle, \quad S' \equiv \theta' v'.$$

From the last two equations of system (2.1), the third moments are expressed in terms of the second moments using the formulas

$$\langle S' v' \rangle \approx -\Phi_1 E^{-1/2} \left[2 \langle v'^2 \rangle \frac{\partial S}{\partial y} + S \frac{\partial \langle v'^2 \rangle}{\partial y} \right]; \quad (2.2)$$

$$\langle \theta'^2 v' \rangle \approx -\Phi_2 E^{-1/2} \left[2S \frac{\partial S}{\partial y} + \langle v'^2 \rangle \frac{\partial \langle \theta'^2 \rangle}{\partial y} \right]. \quad (2.3)$$

After substitution of (2.2), (2.3) into (2.1), we finally obtain a closed system of three quasilinear equations for the three sought stochastic values of the field of the concentration:

$$\left. \begin{aligned} \langle u \rangle \frac{\partial \langle \theta \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle \theta \rangle}{\partial y} + \frac{\partial S}{\partial y} &= 0, \\ \langle u \rangle \frac{\partial S}{\partial x} + \langle v \rangle \frac{\partial S}{\partial y} + \langle v'^2 \rangle \frac{\partial \langle \theta \rangle}{\partial y} &= \Phi_1 \frac{\partial}{\partial y} \left[2 \langle v'^2 \rangle E^{-1/2} \frac{\partial S}{\partial y} + S E^{-1/2} \frac{\partial \langle v'^2 \rangle}{\partial y} \right] - \frac{3}{8} \left[\frac{a_0 + 8b_0}{L} \right] E^{1/2} S - \frac{c_0}{L} E^{1/2} S, \\ \langle u \rangle \frac{\partial \langle \theta'^2 \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle \theta'^2 \rangle}{\partial y} + 2S \frac{\partial \langle \theta \rangle}{\partial y} &= \Phi_2 \frac{\partial}{\partial y} \left[2S E^{-1/2} \frac{\partial S}{\partial y} + \langle v'^2 \rangle E^{-1/2} \frac{\partial \langle \theta'^2 \rangle}{\partial y} \right] - 6b_0 \frac{E^{1/2}}{L} \langle \theta'^2 \rangle, * \end{aligned} \right\} \quad (2.4)$$

where $\Phi_1 \equiv L^{3/4} (a_0 + 4b_0) + c_0)^{-1}$; $\Phi_2 \equiv L^{3/8} (a_0 + 16b_0) + c_0)^{-1}$.

A distinguishing special characteristic of Eqs. (2.4) consists in the fact that they describe a nonlinear diffusion process, in which the diffusion coefficients themselves are the sought quantities.

The closed system of equations for the fields of the velocity and the concentration ((2.4), (1.10) from [4]) has the same characteristic special feature as the closed system of equations of the field of the velocity; it is not a system of the diffusion type; the differential operator with respect to the spatial variable y in the first equation of (2.4) is of the first order. As in [4], it can be demonstrated convincingly [9] that the connected system of the equations of the fields of the velocity and the concentration, in addition to the trivial 11-fold characteristics $x = \text{const}$ ($k_1 \equiv dx : dy = 0$), still has one real characteristic, coinciding with the flow line $k_2 \equiv dx : dy = \langle u \rangle / \langle v \rangle$. Thus, the differential operator of the "connected" system of equations can be related to the weakly hyperbolic type [3].

The boundary-value problem for the system of equations (2.4) demands (in addition to the boundary conditions for the field of the velocity [4]) the assignment of boundary conditions for the field of the concentrations.

$$\begin{aligned} \text{with } y \rightarrow -\infty : \langle \theta \rangle &= 1, \quad S = \langle \theta'^2 \rangle = 0, \\ \text{with } y \rightarrow +\infty : \langle \theta \rangle &= S = \langle \theta'^2 \rangle = 0. \end{aligned} \quad (2.5)$$

With respect to boundary conditions (2.5) the following remark can be made: The field of the concentration in the mixing zone depends not only on the averaged value of the concentration of the additive, but also on the fluctuations of θ with respect to the external flow, moving with a velocity U_0 . Therefore, the conditions $\langle \theta(x, -\infty) \rangle = 1$, $\langle \theta' v' (x, -\infty) \rangle = \langle \theta'^2 (x, -\infty) \rangle = 0$ mean that each element of the liquid in the external flow contains exactly the same concentration of the chemical component and that, therefore, there should be no fluctuations. The fluctuations of the concentrations in the zone of the mixing are then the result of mixing of a flow with a constant mean concentration of an additive in a turbulent field with a gradient of the mean velocity. In this part, the discussion can be improved in what follows by taking into consideration initial fluctuations of the field of the concentration in the external flow, which differ from zero. As initial conditions in the cross section $x = x_0$ there are given the stepwise function for the mean concentration $\langle \theta(x_0, y) \rangle$, and $\langle \theta' v' (x_0, y) \rangle = \langle \theta'^2 (x_0, y) \rangle = 0$.

As for the mean characteristics of the field of the velocity [4], for the mean characteristics of the field of the concentration, under the conditions of the problem, self-similarity is to be expected; this is observed experimentally [5]. This permits making the spatial variables x and y in (2.4) dimensionless, using an arbitrary linear scale, and, comparing the results of a numerical solution with the experimental data, to use the self-similar coordinate $\eta = y/x$ [more exactly, $\eta = y/(x - x_0)$, where x_0 is the arbitrary start of mixing of the homogeneous flow and the surrounding motionless liquid].

*The closed system of equations of the field of the velocity, which must be added to system (2.4), is given in [4] and is not written here.

The calculating scheme for the system of equations of the field of the concentration is constructed in the same way as the difference scheme for the system of equations of the field of the velocity in [4]. All that has been said with respect to the difference scheme in [4] is carried over to the difference scheme for the system (2.4). Since the system (2.4) "connected" with the system of equations of the field of the velocity [4] is "insignificant," the field of the concentration has no effect on the moments of the field of the velocity (as a result of the postulated "passivity" of the additive), an approximate solution (by the method of finite differences) of the system (2.4) can be constructed with a known field of the velocity (and its moments). Under these circumstances, the quality of the difference scheme (in the sense of expenditure of machine time) can be somewhat raised by the use of a more economical method for inversion of the matrix than the method of inversion with selection of the principal element. The essence of the method consists in the fact that, as soon as the matrices of the coefficients of the difference operator must be degenerate over the whole region of integration, it is not necessary to invert the matrix, selecting its principal element each time. The inversion of the matrix can be carried out consecutively, by columns. Thus, for example, in the difference scheme (2.1), (2.2) from [4] (a matrix of the fourth order was inverted) this method of inversion permitted decreasing the calculating time by approximately one and one-half times in comparison with the case where the inversion of the matrix was done with selection of the principal element. With regard to the accuracy of the inversion of the matrix, numerical experiments showed that four significant figures of the inverse matrix coincide with both methods of inversion.

For the numerical calculations, the constant a_0 (a_0/α , where α is the coefficient of proportionality in the expression for the scale of the turbulence $L = \alpha x$ [4]) is determined, as in [4], from experimental data [10] on the dissipation of the energy of the pulsational energy ($a_0/\alpha \cong 6$). The ratio of the "dissipative" $3/4$ (a_0/α) and the "volumetric" c_0/α constants was taken equal to unity. A propos of the comparison made below between the numerical results and the experimental data, it must be noted that the experimental devices in [5, 10, 11] differed somewhat in construction [5], which led to somewhat different mean characteristics of the mixing zone in the self-similar region. This relates to the profile of the mean velocity $\langle u \rangle$ and to a number of second moments of the field of the velocity [10, 11], as well as to the integral scales of the turbulence and the so-called rate of expansion of the mixing zone [12], i.e., to some arbitrary value, making it possible to give a definite idea of the expansion of the turbulent region with increasing distance downstream. In the numerical calculations, this value was found equal to 0.17; according to the data of the experiment of [11], to 0.16, and of the experiment of [10], to 0.20. The constant b_0 (b_0/α) cannot be found a priori, due to the lack of suitable experimental data. To determine the numerical value of this constant, it was necessary to have recourse to its numerical optimization; as a first approximation it was possible to use a rough evaluation, obtained from the following considerations: It can be considered experimentally established that, in a turbulent flow with a gradient of the mean velocity, transfer of a scalar additive takes place more rapidly than the transfer of momentum. With a gradient description of diffusion processes, this difference in the transfer mechanism can be qualitatively characterized by a dimensionless parameter, the turbulent Schmidt number, which must be less than unity. For a plane-parallel flow, from the system of equations (1.10) [4] and the system of equations of the field of the concentration (2.4), neglecting convective and diffusional terms (terms with third moments), we find

$$\langle u'v' \rangle = - \frac{1}{\left[\frac{c_0}{L} + \frac{3}{4} \frac{a_0}{L} \right]} \frac{\langle v'^2 \rangle}{E^{1/2}} \frac{d\langle u \rangle}{dy},$$

$$\langle \theta'v' \rangle = - \frac{1}{\left[\frac{c_0}{L} + \frac{3}{8} \frac{a_0}{L} + \frac{3b_0}{L} \right]} \frac{\langle v'^2 \rangle}{E^{1/2}} \frac{d\langle \theta \rangle}{dy},$$

from which the turbulent Schmidt number

$$Sm = \frac{\left[1 + \left(\frac{3}{8} \frac{a_0}{\alpha} \right) \left(\frac{c_0}{\alpha} \right) + \left(\frac{3b_0}{\alpha} \right) \left(\frac{c_0}{\alpha} \right) \right]}{\left[1 + \left(\frac{3}{4} \frac{a_0}{\alpha} \right) \left(\frac{c_0}{\alpha} \right) \right]} = \frac{3}{4} + \frac{3}{2} \left(\frac{b_0}{\alpha} \right) \left(\frac{c_0}{\alpha} \right) \quad (2.6)$$

(taking account of the fact that the ratio $3/4$ (a_0/α)(c_0/α) is taken equal to unity). From (2.6), with the ratio adopted $3/4$ (a_0/α) = 9/2 [4], we obtain the condition (b_0/α) \leq 3/4. The refined numerical value of b_0/α , found from the condition of the agreement of the calculated profile of the variance of the fluctuations of the concentration $\langle \theta'^2 \rangle$ with experimental value [5], is equal to 0.232. With the numerical implementation of an expanded difference scheme by the method of successive matrix intervals, at the outer boundary of the mixing zone (at " $+\infty$ ") the sought vector must be known (the forward path of the successive intervals is from " $-\infty$ " to " $+\infty$ "), which, in accordance with the boundary conditions at this boundary (2.5) and (1.14)

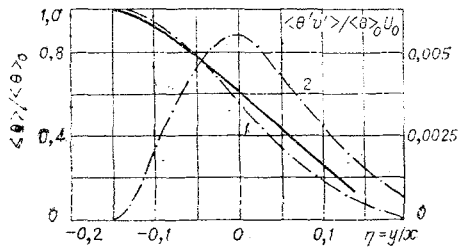


Fig. 1

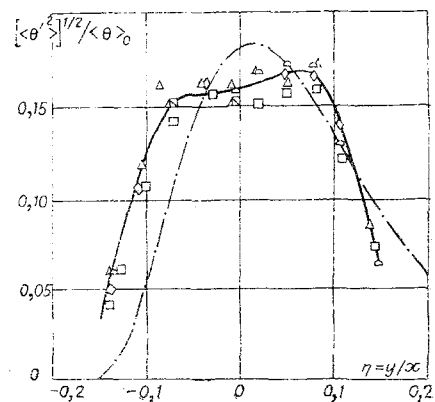


Fig. 2

from [4], is a null path. However, for the system of equations of the boundary-layer type, used to describe the field of the flow in the mixing zone, the homogeneous boundary conditions at the external boundary can obviously not be satisfied if the approximate solution is sought by the method of finite differences. In actuality, the characteristic curve of the connected system of equations for the fields of the velocity and the concentration, coinciding with the line of the flow, has a slope equal to $\arctan(\langle v \rangle / \langle u \rangle)$, which, with $\langle u \rangle \rightarrow 0$ (near the external boundary), takes on an extremal large value. At the limit, with $\langle u \rangle = 0$, it is equal to $\pi/2$; the flow line and the characteristic curve coinciding with it are directed normal to the x axis. The liquid is "drawn into" the turbulent zone from the surrounding quiescent liquid at a right angle and, consequently, $\langle v \rangle \neq 0$. By the same token, near the external boundary, the main premise of the approximation of the boundary layer breaks down: the transverse component of the mean velocity of the flow must be much less than the longitudinal (at the external boundary of the mixing zone, the velocities are quantities of the same order of magnitude). Thus, an attempt to satisfy the boundary condition $\langle u(x, +\infty) \rangle = 0$ lies beyond the limits of the validity of equations of the boundary-layer type, and cannot be correct. It has been found that a stable (without any kind of vibrations) solution near the external boundary can be obtained if, in the sought vector, having seven components, only one of its components $\langle u(x, +\infty) \rangle$, equal to 0.01, is different from zero at " $+\infty$."

Numerical results for the moments of the field of the concentration of a chemically inert additive are given in Figs. 1 and 2, where the solid line is a plot of experimental data [5], and the dashed-dot line is a numerical solution (the "points" of different shape in Fig. 2 correspond to the experimental points of [5]). The number 1 in Fig. 1 denotes the profile of the mean concentration, and the number 2, the transverse component of the turbulent flow of the additive $\langle \theta'v' \rangle$, for which there are no experimental data in [5]. On the whole, the agreement between the numerical solution and the experimental curves can be regarded as fully satisfactory.

3. Effect of a Passive Chemical Reaction on the Field of the Concentration in the Turbulent Mixing Zone

With $\mu \neq 0$, the right-hand part of the system (2.4) contains terms characterizing the rate of a chemical transformation, corresponding to the statistical characteristic curve of the field of the concentration, which, for the kinetic model under consideration, have the form of sinks. After bringing the system (2.4) into dimensionless form by the introduction of characteristic scales of the velocity, the concentration, and the length, in the source terms of the right-hand parts (characterizing the rate of the chemical transformation) there appears a dimensionless parameter, depending on the linear scale l and equal to the ratio of the characteristic time of the turbulent diffusion of the chemical component in the flow l/U_0 to the characteristic time of the chemical reaction $|1/\mu|$

$$\Gamma = (lU_0)/(1/\mu). \quad (3.1)$$

The equation of conservation of the mean concentration of the additive has the form

$$\langle u \rangle \frac{\partial \langle \theta \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle \theta \rangle}{\partial y} + \frac{\partial \langle \theta'v' \rangle}{\partial y} = \Gamma \langle \theta \rangle. \quad (3.2)$$

Since we are considering only one additive, immersed in an incompressible chemically inert liquid, the permissible chemical reaction (here of the first order) will be a reaction of the form

$$\left[\frac{\partial \theta}{\partial t} \right]_{\text{chem}} = -\mu \theta \quad (\mu > 0),$$

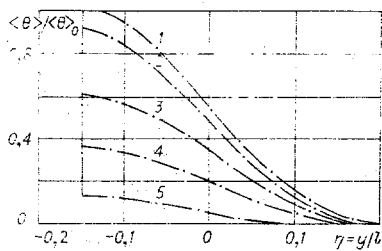


Fig. 3

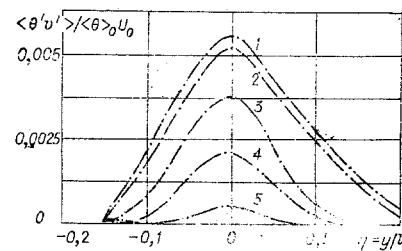


Fig. 4

whose mechanism (with the imposition of the following boundary conditions at the external boundary of the mixing zone with $y \rightarrow -\infty$: $\langle \theta \rangle \rightarrow 1$, $\langle \theta'v' \rangle \rightarrow 0$) must satisfy the condition: a "reaction-rate constant" μ equal to zero if $\langle \theta \rangle = 1$, and having a constant value (different from zero) in the contrary case. The chemical reaction then arises in the mixing zone as the result of the turbulent mixing of the main flow of an incompressible chemically inert liquid (with a chemically active additive immersed in it) with the surrounding (quiescent) medium, playing the role of some "catalyst," inducing a reaction. Since a chemical reaction in a turbulent flow depends on the real time, all of the characteristic scales must be given a priori. The mixing of turbulent flows, with the presence of a chemical reaction in the mixing zone, is considered in an idealized statement [4]; therefore, there is no characteristic linear scale of the problem. This is connected with the fact that, for a description of a chemical reaction in the mixing zone under real conditions, the real initial conditions must also be known, including a knowledge of the characteristic curve of the boundary layer, coming together with the separating wall (see Fig. 1 in [4]). The idealization of the picture of the mixing (in the sense of the assignment of the initial conditions), with a chemical reaction taking place in the flow, does not offer the possibility of assigning the linear scale a priori, which would be attributed to the flow in the mixing zone proper. However, the aim of the consideration of the effect of a chemical reaction of the first order consists in the present case only in taking a qualitative account of the effect of this effect on the moments of the field of the concentration, and fixing the scale l in (3.1) is not obligatory. As the scale l , there can be taken, for example, the height of the slot from which issues the external homogeneous flow (see Fig. 1 in [4]), as this is sometimes done [13]; however, this will be a scale not related to the flow in the mixing zone proper.

In a qualitative consideration of the effect of a chemical reaction, the indeterminacy in assignment of the scale l will be included in the parameter Γ , which can assume constant values, from zero (for the frozen flow of a chemically reacting stream) to minus infinity (the chemically equilibrium case).

With closing of the system of equations of a field of the concentration of the boundary-layer type on the level of third moments, the functions Φ_1 and Φ_2 , appearing here in the corresponding coefficients of the turbulent diffusion, have the form*

$$\begin{aligned} \Phi_1(x, y) &= L \left[\frac{3}{4} (a_0 + 4b_0) + c_0 - \Gamma (L/E^{1/2}) \right]^{-1}, \\ \Phi_2(x, y) &= L \left[\frac{3}{8} (a_0 + 16b_0) + c_0 - 2\Gamma (L/E^{1/2}) \right]^{-1}, \end{aligned} \quad (3.3)$$

and, consequently, the corresponding coefficients of turbulent diffusion are found to be independent of the term characterizing the rate of the chemical transformation. This conclusion was reached earlier in [14], which considered a model problem of turbulent diffusion of a scalar additive in a chemically reacting wake, where, in the equation of the conservation of the concentration, there remained only linear terms of the second order and expressions describing the rate of the chemical reaction. The turbulent field of the velocity was assumed to be given a priori. (The linearization used by the authors of [14] is equivalent to the breaking-off of the chain of moment equations on the level of second moments, neglecting third moments.) An analysis of the problem showed that the term characterizing the turbulent transfer of scalar quantity $\langle \theta'v' \rangle$ is found to depend on the term characterizing the rate of the chemical reaction. Consequently, also the coefficient of turbulent diffusion, if it can be introduced in a chemically reacting flow, will depend on the term characterizing the rate of the chemical transformation.

The numerical results on the effect of a nonequilibrium (of the first order) chemical reaction on the mean concentration of the additive, the transverse component of the turbulent flow of the additive, and the

* The functions Φ_1 and Φ_2 , determined by formulas (3.3) must stand under the signs of the derivative with respect to y in the right-hand parts of the last two equations of the system (2.4).

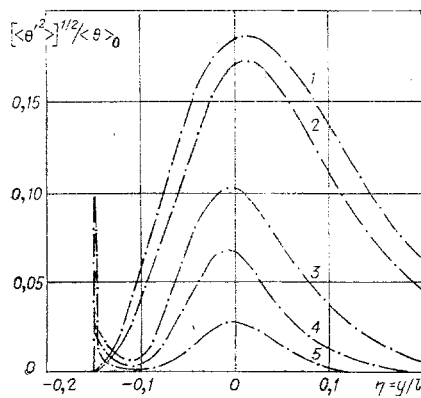


Fig. 5

variance of the fluctuations of the concentration of the additive are shown in Figs. 3-5. In these figures, the curves correspond to different values of the parameter Γ : 1) $\Gamma = 0$; 2) $\Gamma = -0.08$; 3) $\Gamma = -0.5$; 4) $\Gamma = -0.1 \cdot 10^4$; 5) $\Gamma = 0.2 \cdot 10^4$. Along the axis of abscissas on Figs. 3-5 there is plotted the dimensionless transverse coordinate y/l , while the curves given correspond, with respect to the coordinate x , to the region where there is self-similarity of the statistical properties of the fields of the velocity and the concentration in the absence of a chemical reaction (with $\Gamma = 0$). A chemical reaction, with a rise (in absolute value) of the parameter Γ , leads to a decrease in the amplitude values $\langle \theta'v' \rangle$, $\langle \theta'^2 \rangle$ and of the mean concentration $\langle \theta \rangle$; the character of the decrease at the external boundary of the mixing zone (at " $-\infty$ ") is found to be not smooth, but stepwise. This is a consequence of the boundary conditions imposed at this boundary, and of the special characteristics of the course of the reaction in the mixing zone pointed out above.

For $\Gamma \neq 0$ (see Fig. 3) the profile of the mean concentration takes on an almost "stepwise" form, and the size of the "steps" rises with a rise in the absolute value of the parameter Γ . At the equilibrium limit with $\Gamma \rightarrow -\infty$, $\langle \theta \rangle \rightarrow 0$, the distribution of $\langle \theta \rangle$ in a transverse direction has the form of a stepwise function. Such a character of the decrease in the function $\langle \theta \rangle$ manifests itself in the profile of $[\langle \theta'^2 \rangle]^{1/2}$ (see Fig. 5), bringing about a rise in the function $[\langle \theta'^2 \rangle]^{1/2}$ with an increase in the absolute value of the parameter Γ near the external boundary of the mixing zone, due to a rise in the gradient $\partial \langle \theta \rangle / \partial y$ (the term $2 \langle \theta'v' \rangle$ ($\partial \langle \theta \rangle / \partial y$) itself characterizes the generation of $\langle \theta'^2 \rangle$ as a result of the interaction between the turbulent pulsations and the gradient of the mean concentration). Analogous behavior is observed in the decrease in the turbulent flow of the additive (see Fig. 4) with a rise in the absolute value of the parameter Γ . However, the behavior of the function $\langle \theta'v' \rangle$ near the external boundary of the mixing zone, with conservation of the trend in the behavior of the function $\langle \theta'^2 \rangle$ near this boundary, is less sharply expressed.

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TURBULENT HEAT TRANSFER IN A FLOW OF LIQUID
METAL NEAR THE WALL

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UDC 532.517.4 +536.2.023

The article discusses turbulent heat transfer in media with small Prandtl numbers ($Pr \ll 1$ for liquid metals). In this case, the thermal sublayer is Pr^{-1} times thicker than the viscous sublayer. It is established that the coefficient of turbulent heat transfer varies in the thermal sublayer proportionally to the second power of the distance to the wall; the ratio of the coefficients of the turbulent transfer of heat and momentum in this region decreases in accordance with a linear law with approach to the wall. The conclusions of the theory are compared with the experimental data of other authors.

As is well known, the Prandtl numbers for liquid metals are small: $Pr = \nu/\chi \sim 10^{-2} \dots 10^{-3}$ (ν is the kinematic viscosity; χ is the coefficient of thermal diffusivity), and, with $\chi \gg \nu$, the region of an influence of the molecular effects of heat transfer ("the thermal sublayer") is far larger than the viscous sublayer, whose dimensions are determined by the scale $y_1 = \nu/v_*$ ($v_* = \sqrt{\tau_w/\rho}$ is the parameter of the "dynamic velocity"; τ_w is the friction stress at the wall; ρ is the density of the liquid). The thickness of the thermal sublayer with $Pr \ll 1$ is determined by the scale $y_2 = \chi/v_*$ [1]. Outside the thermal sublayer, in the layer of constant friction stress, considerations of dimensionalities give a value of $\chi_T(y) = \text{const } v_* y$, where $\chi_T(y)$ is the coefficient of turbulent thermal diffusivity; y is the distance to the wall. The behavior of the function $\chi_T(y)$ in the thermal sublayer is determined in accordance with the equation for the pulsations of the temperature.

1. Let us consider the turbulent flow of an incompressible liquid, flowing above a smooth surface in the direction of the x axis; we direct the y axis along a normal to the wall; the z axis is perpendicular to the x and y axes. We denote by $U(y)$ the mean velocity of the flow, and by u, v, w the pulsational components of the velocity in the x, y, z directions, respectively. The turbulence is assumed to be statistically steady-state with respect to the time and homogeneous with respect to the coordinates x and z .

We limit ourselves to a consideration of the region near the wall $y \ll L$ (L is the external scale of the flow), where the turbulence has a universal character [1, 2]. The basic premises of the theory of the similarity of flow near the wall are formulated in the form of two hypotheses, analogous to the Kolmogorskii similarity hypotheses [1]:

1. In the case of turbulence near the wall with sufficiently large Reynolds numbers Re , the statistical conditions of turbulence of the pulsations of the velocity in a region located close to a smooth wall are uniquely determined by two parameters: v_* and ν .

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